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Formulas are derived for the bias in the maximum likelihood estimators (MLE) of the item parameters in the logistic item response model when examinee abilities are known. Numerical results are given for a typical verbal test for college admission. Most typically the bias of an MLE is about one-tenth of its standard error. It is very seldom more than a fifth of its standard error. The effect of bias for individual item-parameter estimates is thus probably negligible. (Author/PN)

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Frederic M. Lord

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Frederic M. Lord, Principal Investigator

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Princeton, New Jersey

April 1982

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Abstract

Formulas are derived for the bias in the maximum likelihood estimators of the item parameters in the logistic item response model when examinee abilities are known. Numerical results are given for a typical verbal test for college admission.

Statistical Bias in Maximum Likelihood Estimators of Item Parameters*

This paper derives formulas for the statistical bias in the maximum likelihood estimators (MLE) of item parameters in item response theory (IRT) [Lord, 1980]. It will deal only with the three-parameter logistic model for dichotomously scored items. Available formulas for the sampling variance of these MLE are limited to the case where the examinee parameters are known; the present derivations are limited to this case also. Under the three-parameter logistic model, the probability of a correct answer to an item is the following function of examinee ability level θ :

$$P \equiv P(\theta) \equiv c + \frac{1 - c}{1 + e^{-A\theta - B}} \equiv 1 - \frac{1 - c}{1 + e^{A\theta + B}} \qquad (1)$$

where A , B , and c are parameters describing the item.

In practical work, $\hat{A}$, the MLE of A , sometimes tends to become infinite. This suggests a positive bias, at least in some data for certain items. Is it possible to correct for this sometimes substantial bias in $\hat{A}$? Practical experience also suggests that

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when B is large and positive, $\hat{B}$ tends to be positively biased; when B is large and negative, $\hat{B}$ tends to be negatively biased. Practical experience has not provided any clear indications as to the bias in $\hat{c}$. Since $\hat{c}$ values are most often less than the reciprocal of the number of choices in a multiple-choice item, it is of interest to determine whether this apparent anomaly could be due to a substantial negative bias in $\hat{c}$.

The method to be used here to derive formulas for the bias in estimated item parameters is the same method described in Lord [Note 1]. The reader is referred there for a more detailed discussion. The following derivation deals with a single fixed item administered to N examinees with known ability levels $\theta_1, \theta_2, \ldots, \theta_N$.

1. Likelihood Equations

Let α denote either A, B, or c. We assume, as in Lord [Note 1] that A, B, and $\hat{\alpha}_a$ are bounded and that c is bounded away from 1. Under these conditions, $\hat{\alpha}$ is a consistent estimator of α and $\sqrt{N}(\hat{\alpha} - \alpha)$ is asymptotically normally distributed with mean zero and finite variance. It follows that $\delta(\hat{\alpha} - \alpha)^s$ is at most of order $N^{-s/2}$.

Let $u_a = 0$ or 1 denote the score of examinee a ($a = 1, 2, \ldots, N$) on the given (dichotomously scored) item. Write $P_a \equiv P(\theta_a)$ and $Q_a \equiv 1 - P_a$. For $a = 1, 2, \ldots, N$ and $\alpha = A, B, c$, write

$$P'^{a}_{\alpha} \equiv \frac{\partial P_a}{\partial \alpha} \quad , \tag{2}$$

$$R'^{a}_{\alpha} \equiv \frac{P'^{a}_{\alpha}}{P_a Q_a} \quad , \tag{3}$$

$$\Gamma^{a}_{\alpha} \equiv (u_a - P_a) R'^{a}_{\alpha} \quad , \tag{4}$$

$$L_{\alpha} \equiv \sum_{a=1}^{N} \Gamma^{a}_{\alpha} \quad . \tag{5}$$

The likelihood equations are now

$$\hat{L}_{\alpha} \equiv 0 \qquad (\alpha = A,B,c) \tag{6}$$

where the caret denotes substitution of MLE for A , B , and c .

2. Taylor Series

Let the symbols β (= A,B,c) and δ (= A,B,c) have the same meaning as α , so that $\sum_{\beta}$ denotes a three-term sum with β taking on the values A , B , and c . Expanding (6) in a three-variable Taylor series and dividing by N , we have

$$0 = \frac{1}{N} \hat{L}_{\alpha} = \frac{1}{N} \sum_{a} \left[\Gamma^{a}_{\alpha} + \sum_{\beta} (\hat{\beta} - \beta) \Gamma^{a}_{\alpha\beta} \right.$$

$$\left. + \frac{1}{2} \sum_{\beta} \sum_{\delta} (\hat{\beta} - \beta)(\hat{\delta} - \delta) \Gamma^{a}_{\alpha\beta\delta} + \ldots \right] \qquad (\alpha = A, B, c) \tag{7}$$

where

$$\left.\begin{aligned} \Gamma^a_{\alpha\beta} &\equiv \partial\, \Gamma^a_\alpha / \partial\, \beta \\ \Gamma^a_{\alpha\beta\delta} &\equiv \partial^2 \Gamma^a_\alpha / \partial\beta\partial\delta \end{aligned}\right\} \quad . \tag{8}$$

For simplicity, write v , w , x , y , z , or Z instead of $\hat{\beta} - \beta$ or $\hat{\delta} - \delta$. It will not be confusing to replace subscripts α , β , δ by v , w , x , y , or z . The Taylor series is now

$$0 = \frac{1}{N} \sum_a \left[\Gamma^a_x + \sum_y y\Gamma^a_{xy} + \frac{1}{2} \sum_y \sum_z yz\Gamma^a_{xyz} + \ldots\right] \quad . \tag{9}$$

Define

$$\gamma^a \equiv \delta\Gamma^a \quad , \qquad \varepsilon^a \equiv \Gamma^a - \gamma^a \quad . \tag{10}$$

It can be seen from (4) that

$$\gamma^a_x = 0 \qquad (x = A,B,c) \quad . \tag{11}$$

Equation (9) can now be written

$$0 = \frac{1}{N} \sum_a \left[\varepsilon^a_x + \sum_y y\gamma^a_{xy} + \sum_y y\varepsilon^a_{xy} + \frac{1}{2} \sum_y \sum_z yz(\gamma^a_{xyz} + \varepsilon^a_{xyz}) + \ldots\right] \quad .$$

Let

$$\gamma_x \equiv \frac{1}{N} \sum_a \gamma^a_x \quad , \qquad \varepsilon_x \equiv \frac{1}{N} \sum_a \varepsilon^a_x \quad , \qquad \varepsilon_{xy} = \frac{1}{N} \sum_a \varepsilon^a_{xy} \quad , \tag{13}$$

and so forth. The Taylor series is now

$$0 = \epsilon_x + \sum_y y\gamma_{xy} + \sum_y y\epsilon_{xy} + \frac{1}{2}\sum_y\sum_z yz(\gamma_{xyz} + \epsilon_{xyz}) + \ldots \quad . (14)$$

Rewrite this in matrix notation,

$$\Gamma \underset{\sim}{y} = -\underset{\sim}{\epsilon} - E\underset{\sim}{y} - M\underset{\sim}{y} - H\underset{\sim}{y} - \ldots \qquad (15)$$

where $\Gamma \equiv \|\gamma_{xy}\|$, $\underset{\sim}{y} \equiv \{\hat{A} - A, \hat{B} - B, \hat{c} - c\}'$, $\underset{\sim}{\epsilon} \equiv \{\epsilon_x\}'$, $E \equiv \|\epsilon_{xy}\|$, $M \equiv \|\frac{1}{2}\sum_z z\gamma_{xyz}\|$, and $H \equiv \|\frac{1}{2}\sum_z z\epsilon_{xyz}\|$. Premultiply (15) by Γ^{-1} to obtain finally

$$\underset{\sim}{y} = -\Gamma^{-1}\underset{\sim}{\epsilon} - \Gamma^{-1}E\underset{\sim}{y} - \Gamma^{-1}M\underset{\sim}{y} - \Gamma^{-1}H\underset{\sim}{y} - \ldots \quad . \qquad (16)$$

The expectation of (16) gives the bias of the vector $\underset{\sim}{y}$ of maximum likelihood estimators. First we will need to eliminate $\underset{\sim}{y}$ from the right side of (16).

3. Solving for $\underset{\sim}{y}$

Premultiply (16) by $\Gamma^{-1}H$ to obtain

$$\Gamma^{-1}H\underset{\sim}{y} = -\Gamma^{-1}H\Gamma^{-1}\underset{\sim}{\epsilon} - \ldots \quad . \qquad (17)$$

In Section 5 it will become clear that the higher-order terms in (17) can be neglected. Equation (17) allows us to eliminate $\underset{\sim}{y}$ from the last term in (16).

Similarly, to evaluate the next-to-last term in (16), premultiply (16) by $\Gamma^{-1}M$, obtaining

$$\Gamma^{-1}M\underset{\sim}{y} = -\Gamma^{-1}M\Gamma^{-1}\underset{\sim}{\varepsilon} - \ldots \quad . \tag{18}$$

Likewise, premultiply (16) by $\Gamma^{-1}E$ to obtain

$$\Gamma^{-1}E\underset{\sim}{y} = -\Gamma^{-1}E\Gamma^{-1}\underset{\sim}{\varepsilon} - \ldots \quad . \tag{19}$$

We eliminate $\underset{\sim}{y}$ from the right-hand side of (16) by substituting (17) - (19) into (16):

$$\underset{\sim}{y} = -\Gamma^{-1}\underset{\sim}{\varepsilon} + \Gamma^{-1}(E + M + H)\Gamma^{-1}\underset{\sim}{\varepsilon} + \ldots \quad .$$

In scalar notation, this is

$$y = -\sum_x \gamma^{yx}\varepsilon_x + \sum_v \sum_w \sum_x \gamma^{yv}(\varepsilon_{vw} + m_{vw} + h_{vw})\gamma^{wx}\varepsilon_x + \ldots \tag{20}$$

where $\|\gamma^{yx}\|^{-1} \equiv \|\gamma_{yx}\|$, $m_{vw} \equiv \frac{1}{2}\sum_y \gamma\gamma_{vwy}$, and $h_{vw} \equiv \frac{1}{2}\sum_y \gamma\varepsilon_{vwy}$.

To evaluate m_{vw}, multiply (20) by $\frac{1}{2}\gamma_{vwy}$ and sum over y to obtain

$$m_{vw} \equiv -\frac{1}{2}\sum_y \sum_x \gamma_{vwy}\gamma^{yx}\varepsilon_x + \ldots \quad . \tag{21}$$

For h_{vw}, similarly

$$h_{vw} = -\frac{1}{2}\sum_y \sum_x \varepsilon_{vwy}\gamma^{yx}\varepsilon_x + \ldots \quad . \tag{22}$$

Substituting (21) and (22) into (20), we have

$$y = -\sum_x \gamma^{yx}\varepsilon_x + \sum_v\sum_w\sum_x \gamma^{yv}\varepsilon_{vw}\gamma^{wx}\varepsilon_x - \frac{1}{2}\sum_v\sum_w\sum_x\sum_z\sum_Z \gamma^{yv}\gamma_{vwz}\gamma^{zZ}\varepsilon_Z\gamma^{wx}\varepsilon_x$$

$$- \frac{1}{2}\sum_v\sum_w\sum_x\sum_z\sum_Z \gamma^{yv}\varepsilon_{vwz}\gamma^{zZ}\varepsilon_Z\gamma^{wx}\varepsilon_x + \cdots \quad . \tag{23}$$

The bias of $\hat{A}$, $\hat{B}$, or $\hat{c}$ is found by taking the expectation of (23). First we need formulas for various derivatives that appear in (23).

4. Derivatives

From (8) and (4)

$$\Gamma^a_{\alpha\beta} = u_a R''^a_{\alpha\beta} - P'^a_\beta R'^a_\alpha - P_a R''^a_{\alpha\beta} \tag{24}$$

where $R''^a_{\alpha\beta}$ denotes a second derivative. From (24)

$$\Gamma^a_{\alpha\beta\delta} = u_a R'''^a_{\alpha\beta\delta} - P''^a_{\beta\delta}R'^a_\alpha - P'^a_\beta R''^a_{\alpha\delta} - P'^a_\delta R''^a_{\alpha\beta} - P_a R'''^a_{\alpha\beta\delta} \quad . \tag{25}$$

To evaluate (24) and (25), various derivatives of (1) are required. Dropping the affix a, we find

$$P'_c = \frac{1}{1 + e^{A\theta+B}} = \frac{Q}{1 - c} \quad , \tag{26}$$

$$P'_B = \frac{Q(P - c)}{1 - c} = (P - c)P'_c \quad , \tag{27}$$

$$P'_A = \theta P'_B \quad , \tag{28}$$

$$P''_{cc} = 0 \quad , \tag{29}$$

$$P''_{Bc} = \frac{-P'_B}{1 - c} \quad , \tag{30}$$

$$P'_{BB} = \frac{P'_B}{1 - c} (1 - 2P + c) \quad , \tag{31}$$

$$P''_{Ac} = - \frac{P'_A}{1 - c} \quad , \tag{32}$$

$$P''_{AB} = \theta P''_{BB} \quad , \tag{33}$$

$$P''_{AA} = \theta P''_{AB} \quad . \tag{34}$$

5. Expectations

Since $\delta u = P$,

$$\gamma_x \equiv \delta \frac{1}{N} \sum_a \Gamma_x^a = 0 \tag{35}$$

From (24) - (25),

$$\gamma_{xy} = -\frac{1}{N}\sum_a P'^a_y R'^a_x = -\frac{1}{N}\sum_a \frac{P'^a_x P'^a_y}{P_a Q_a} \quad , \tag{36}$$

$$\gamma_{xyz} = -\frac{1}{N}\sum_a (P''^a_{yz} R'^a_x + P'^a_y R''^a_{xz} + P'^a_z R''^a_{xy})$$

$$= \frac{1}{N}\sum_a [2P'^a_x P'^a_y P'^a_z \frac{Q_a - P_a}{P_a^2 Q_a^2} - \frac{P'^a_z P''^a_{xy}}{P_a Q_a} - \frac{P'^a_x P''^a_{yz}}{P_a Q_a} - \frac{P'^a_y P''^a_{xz}}{P_a Q_a}] \quad . \tag{37}$$

Writing

$$t_a \equiv \frac{u_a - P_a}{P_a Q_a} \quad ,$$

we now have

$$\epsilon_x = \frac{1}{N}\sum_a t_a P'^a_x \quad , \tag{38}$$

$$\epsilon_{xy} = \frac{1}{N}\sum_a (u_a - P_a) R''^a_{xy} = \frac{1}{N}\sum_a t_a P''^a_{xy} + \frac{1}{N}\sum_a P'^a_x P'^a_y \left(\frac{1}{P_a Q_a} - t_a^2\right) \quad . \tag{39}$$

To evaluate (23) we need

$$\delta\epsilon_x = 0 \quad , \tag{40}$$

$$\delta\epsilon_x \epsilon_z = \frac{1}{N^2}\sum_a \sum_b P'^a_x P'^b_z \, \mathrm{Cov}(t_a, t_b) = \frac{1}{N^2}\sum_a P'^a_x P'^a_z / P_a Q_a$$

$$= -\frac{1}{N}\gamma_{xz} \quad . \tag{41}$$

This last result is obtained from (38), using the fact that $\mathrm{Cov}(t_a, t_b)$ is zero when $a \neq b$ and that $\mathrm{Var}\, t_a = 1/P_a Q_a$. Similarly, from (38) and (39), we find that

$$\delta\varepsilon_{vw}\varepsilon_x = \frac{1}{N^2}\sum_a\sum_b P''^a_{vw}P'^b_x \mathrm{Cov}(t_a,t_b) - \frac{1}{N^2}\sum_a\sum_b P'^a_v P'^a_w P'^b_x \mathrm{Cov}(t_a,t_b^2)$$

$$= \frac{1}{N^2}\sum_a \left(\frac{P''^a_{vw}P'^a_x}{P_aQ_a} + P'^a_v P'^a_w P'^a_x \frac{P_a - Q_a}{P_a^2 Q_a^2} \right) . \qquad (42)$$

Also, from (25) and (37),

$$\delta\varepsilon_{vwz}\varepsilon_z\varepsilon_x$$

$$= \frac{1}{N^3}\sum_a\sum_b\sum_c (u_a - P_a)R'''^a_{vwz}(u_b - P_b)\frac{P'^b_z}{P_bQ_b}(u_c - P_c)\frac{P'^c_x}{P_cQ_c} . \qquad (43)$$

Since $\delta(u_a - P_a)(u_b - P_b)(u_c - P_c)$ vanishes unless $a = b = c$, (43) is of order $1/N^2$ and can be neglected in evaluating the expectation of (23). The order of magnitude of other terms neglected in preceding sections can be found by the same method.

6. Bias in Item Parameter MLE

The bias in the MLE of an item parameter is now found by writing down the expectation of (23), dropping the last term, and evaluating the remaining terms on the right by (40), (41), and (42). The resulting formula for the bias, accurate through terms of order $1/N$, is

$$y \doteq \sum_v \sum_w \sum_x \gamma^{yv}\gamma^{wx}\delta\varepsilon_{vw}\varepsilon_x + \frac{1}{2N}\sum_v \sum_w \sum_x \sum_z \gamma^{yv}\gamma_{vwz}\gamma^{wx}\sum_Z \gamma^{zZ}\gamma_{xZ}$$

$$\doteq \sum_v \gamma^{yv}\sum_w \sum_x \gamma^{wx}\delta\varepsilon_{vw}\varepsilon_x + \frac{1}{2N}\sum_v \gamma^{yv}\sum_w \sum_x \gamma^{wx}\gamma_{vwx} \qquad (44)$$

(since the sum over Z equals 1 when $z = x$ and vanishes otherwise). The terms on the right are evaluated using (36), (37), and (42). The y on the left side of (44) is either $\hat{A} - A$, $\hat{B} - B$, or $\hat{c} - c$. The affixes on the right side denote either A, B, or c; $\|\gamma^{yv}\|$ denotes the inverse of $\|\gamma_{vy}\|$.

7. Reparameterization

The preceding sections derive the bias of $\hat{A}$ and $\hat{B}$ (for convenience), whereas the item parameters commonly used are $a = A/1.7$ and $b = -B/1.7a$. The bias of $\hat{a}$ is clearly equal to the bias of $\hat{A}$ divided by 1.7. The bias of $\hat{b}$ may be found as follows. $\mathrm{Cov}(\hat{a},\hat{b}) =$

$\delta(\hat{a} - a)(\hat{b} - b) \equiv \delta(\hat{a}\hat{b} - ab) - b\delta(\hat{a} - a) - a\delta(\hat{b} - b)$. Rearranging this identity, we find the bias of $\hat{b}$:

$$\delta(\hat{b} - b) \equiv \frac{1}{A} [-\delta(\hat{B} - B) - b\delta(\hat{A} - A) - 1.7\ \mathrm{Cov}(\hat{a},\hat{b})] \quad . \qquad (45)$$

The required covariance on the right is obtained in the usual way, by inverting the information matrix [Lord, 1980, p. 191].

8. Numerical Example

Figures 1, 2, 4, 6 show the bias in $\hat{b}$, $\hat{a}$, and $\hat{c}$ for a set of 90 items selected to represent very roughly a typical verbal test for college admissions. This is artificial data, thus the true parameters are known. The number of examinees used to estimate the item parameters is 2995.

Because of the limitations of three-dimensional plotting, Figure 1 shows only those items for which $\hat{b}$ is positively biased; Figure 2 shows the remaining items. Easy and medium-diffiulty items are negatively biased; only difficult items are positively biased. Items with $b = 1.5$ to 1.8 have near-zero bias. For five items the bias is so large that it runs off the plot. The item parameters and biases for these five items are as follows:

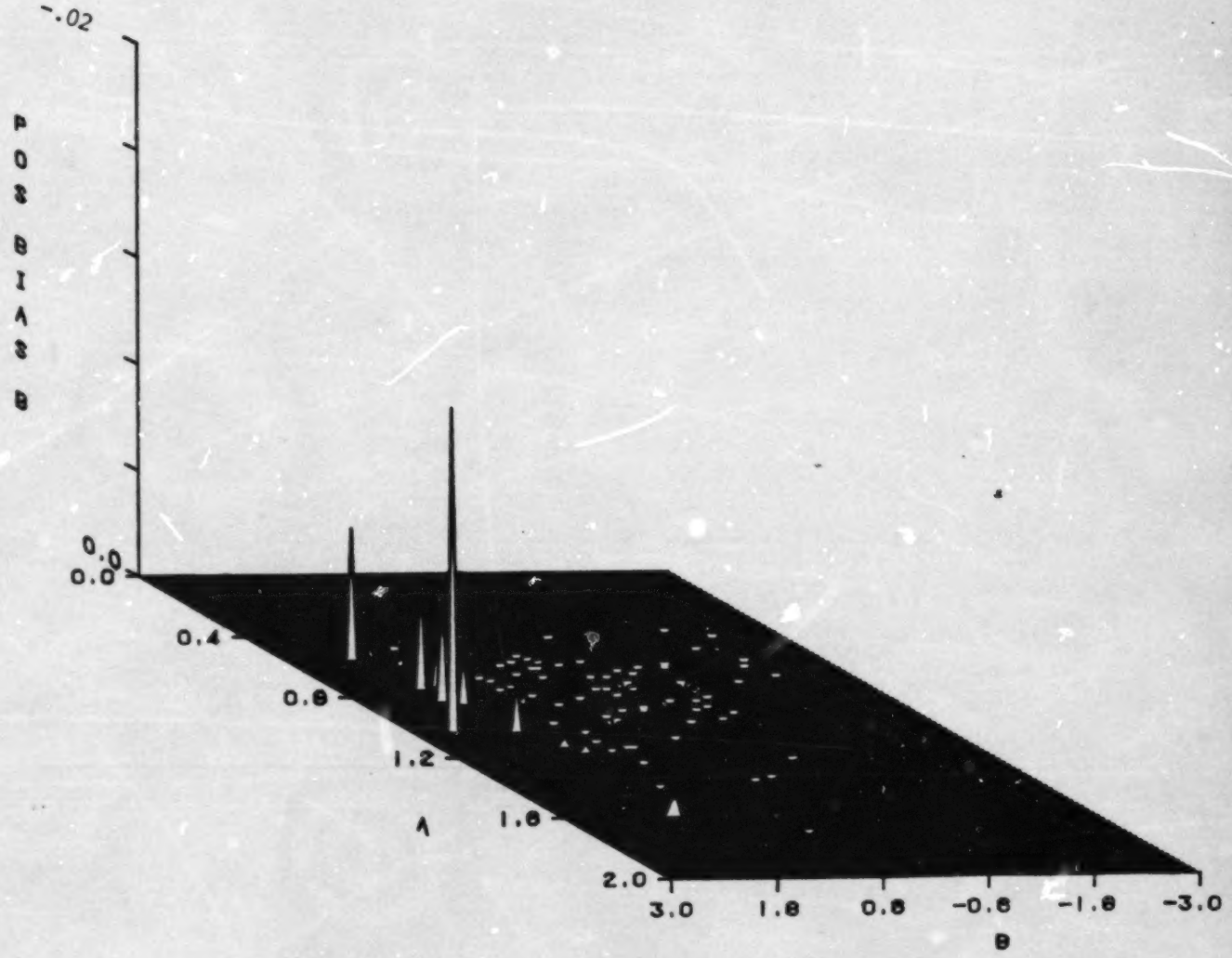

Figure 1. Statistical bias in the maximum likelihood estimator $\hat{b}$ for items with $s\hat{b} > b$.

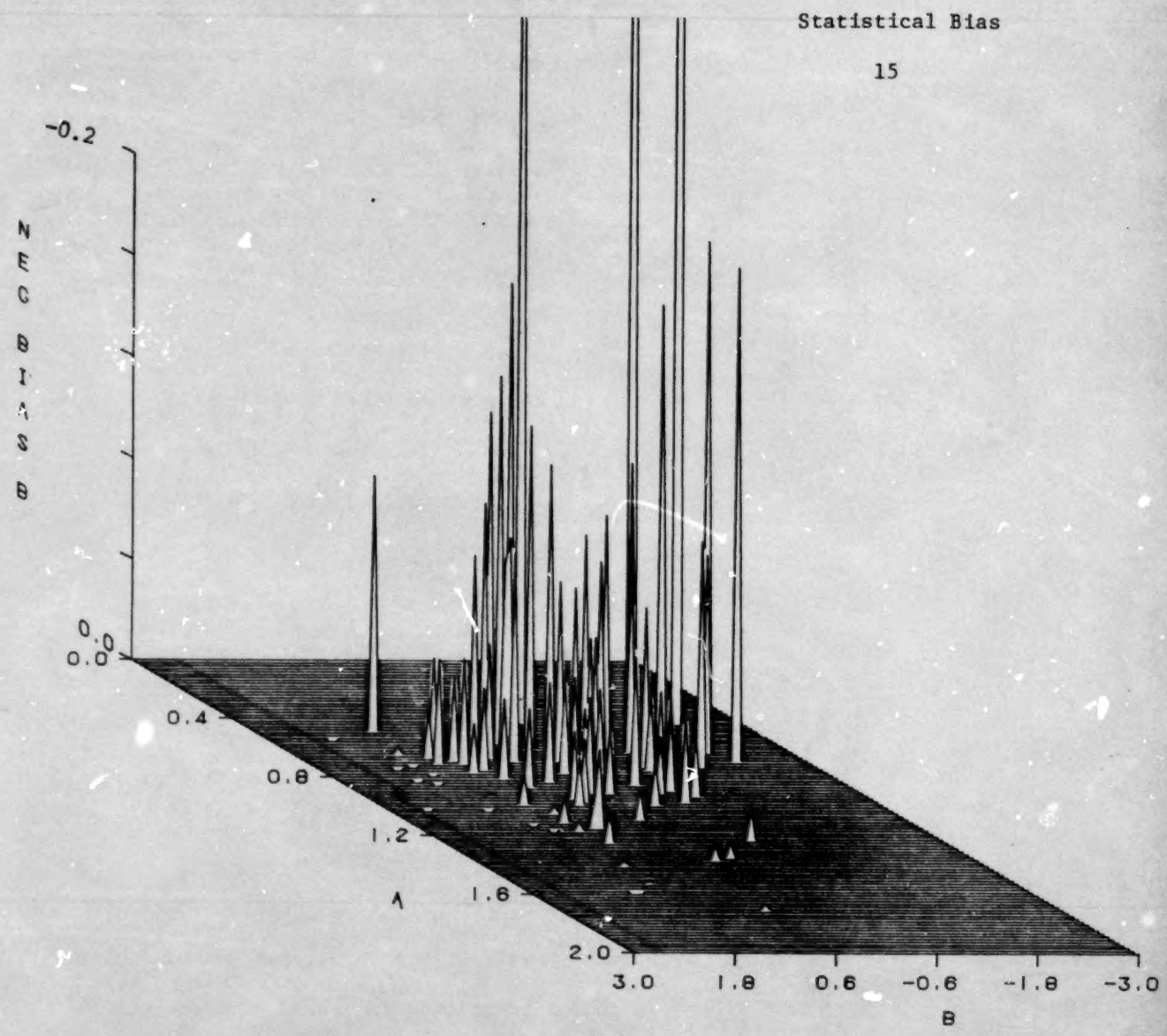

Figure 2. Bias in $\hat{b}$ for items with $\hat{b} < b$.

	b	c	$\delta(\hat{b}-b)$	$\delta(\hat{a}-a)$	$\delta(\hat{c}-c)$
.44	-4.4	.01	-.37	.009	-.43
.43	-3.1	.10	-.11	.006	-.08
.41	-2.2	.15	-.08	.005	-.04
.37	-1.8	.01	-.07	.003	-.03
.41	-0.4	.17	-.05	.004	-.01

The first two of these five items do not appear in the plots at all because the b values lie outside the plotted range. We see that low discriminating power and low difficulty (high easiness) give rise to large estimation errors for $\hat{b}$, as might be expected.

Figures 3, 5, 7 show the standard errors of $\hat{b}$, $\hat{a}$, and $\hat{c}$ for comparison. For clarity the a and b scales are oriented one way in Figures 1-3 and 6-7, the opposite way in Figures 4-5. Note that the vertical scales vary from figure to figure.

The bias in $\hat{a}$ is positive for all items, the bias in $\hat{c}$ is negative for all items. In general, an estimate that has a large standard error tends to have a numerically large bias.

For very easy items, $\hat{b}$ and $\hat{c}$ have numerically large biases and large standard errors. For hard items, $\hat{c}$ has a numerically small bias and small standard error, $\hat{a}$ has a large bias and large standard error. The bias and standard error of $\hat{b}$ both increase for very difficult items. Highly discriminating items have numerically small bias and small standard error for $\hat{b}$ and $\hat{c}$. Poorly discriminating items tend to have low bias and low standard error for $\hat{a}$.

The plots show the relation of bias (or of standard error) to a and b. The relation to c is not easily made graphically

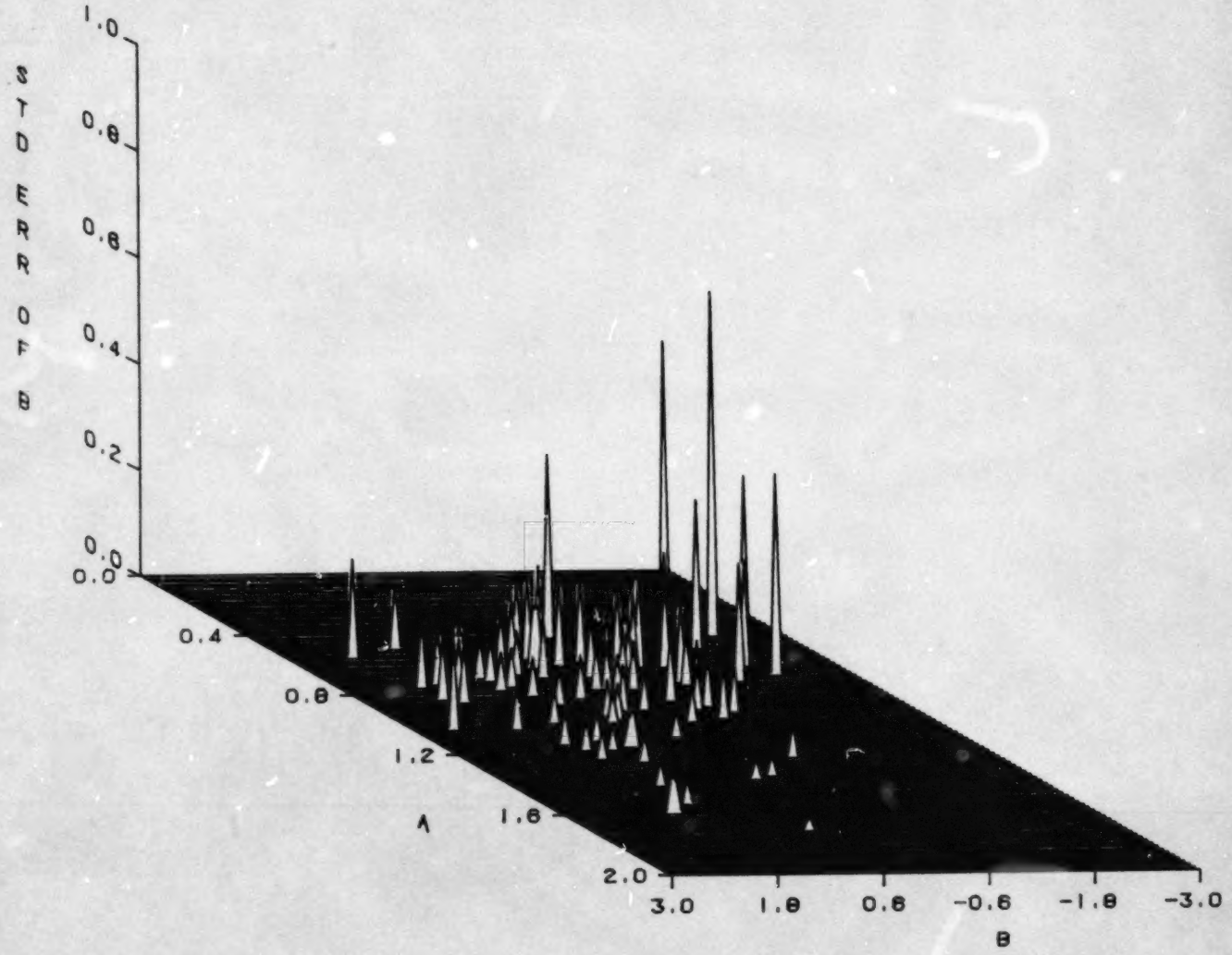

Figure 3. Standard error of $\hat{b}$ for all items.

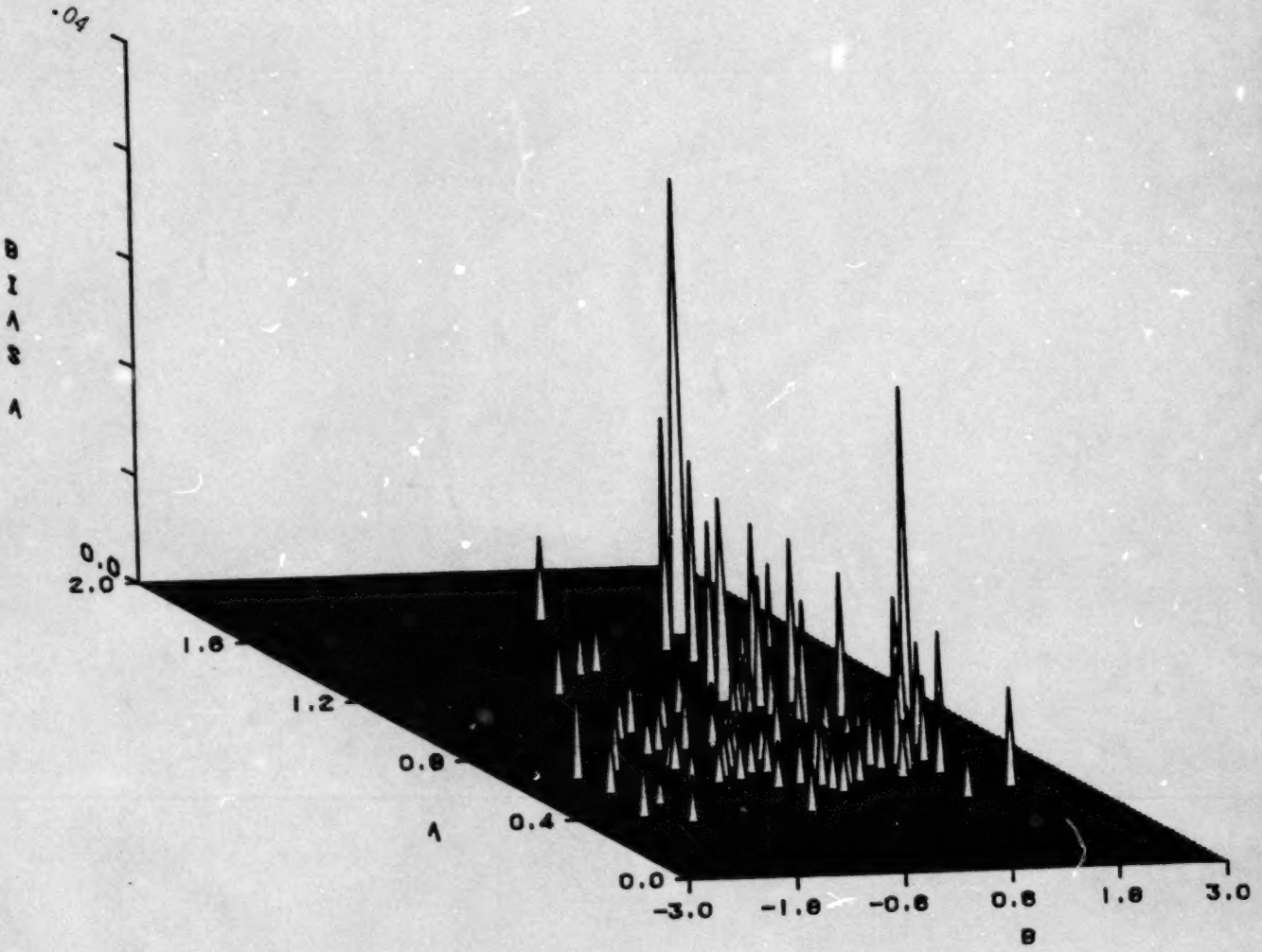

Figure 4. Bias in $\hat{a}$.

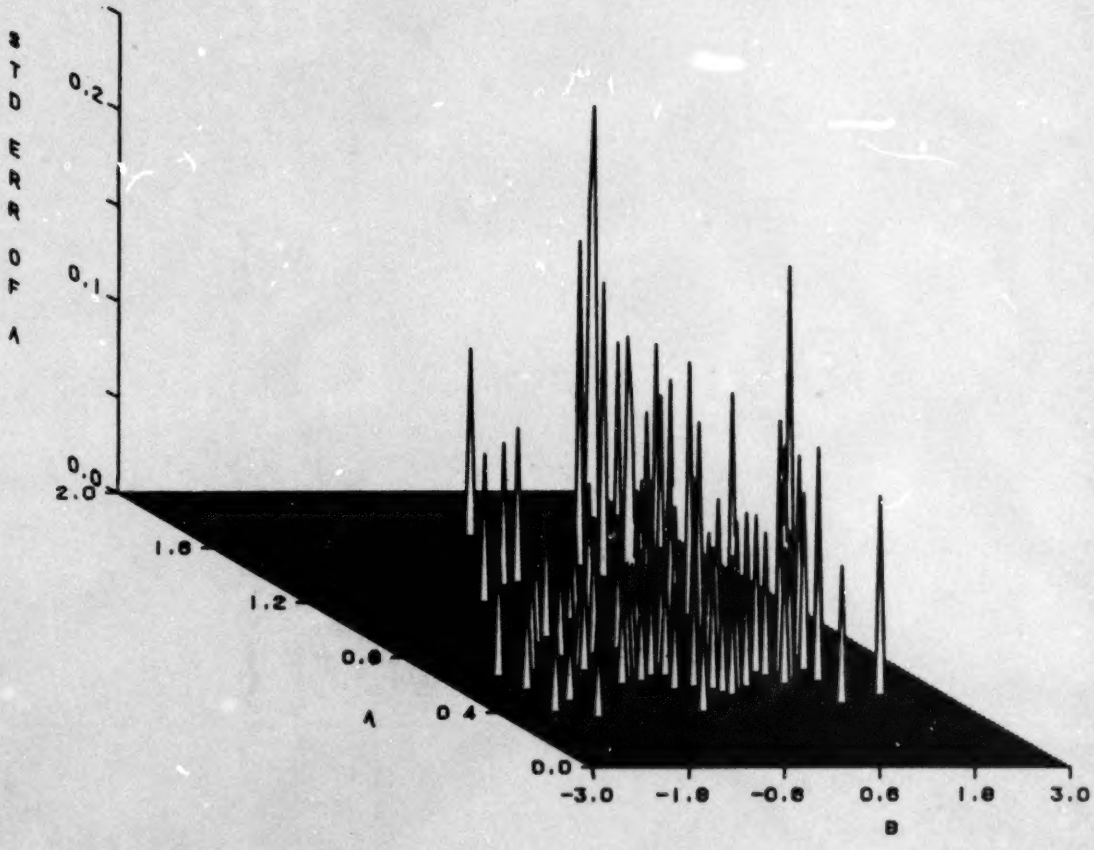

Figure 5. Standard error of $\hat{a}$.

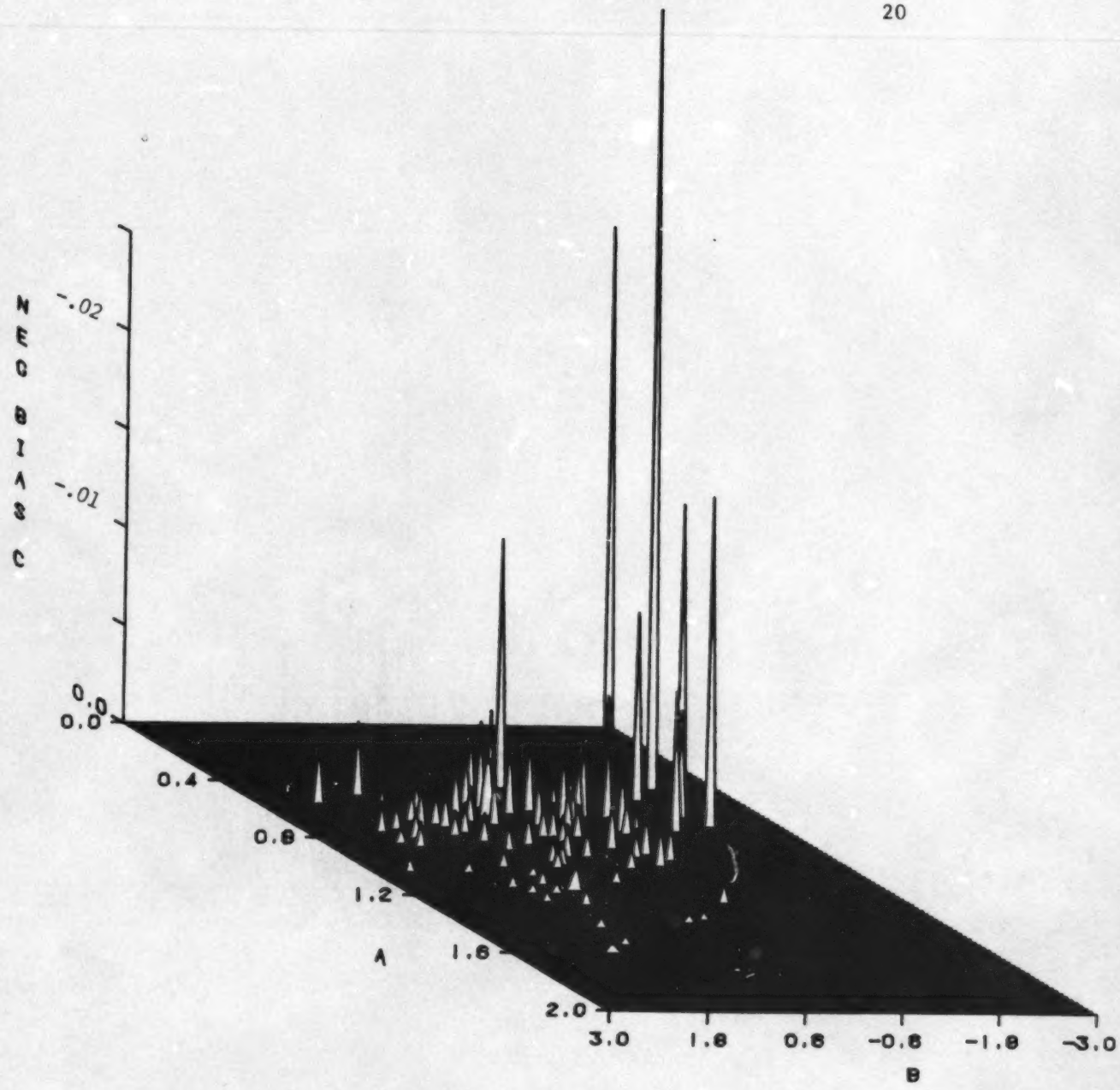

Figure 6. Bias in $\hat{c}$ (all biases are negative).

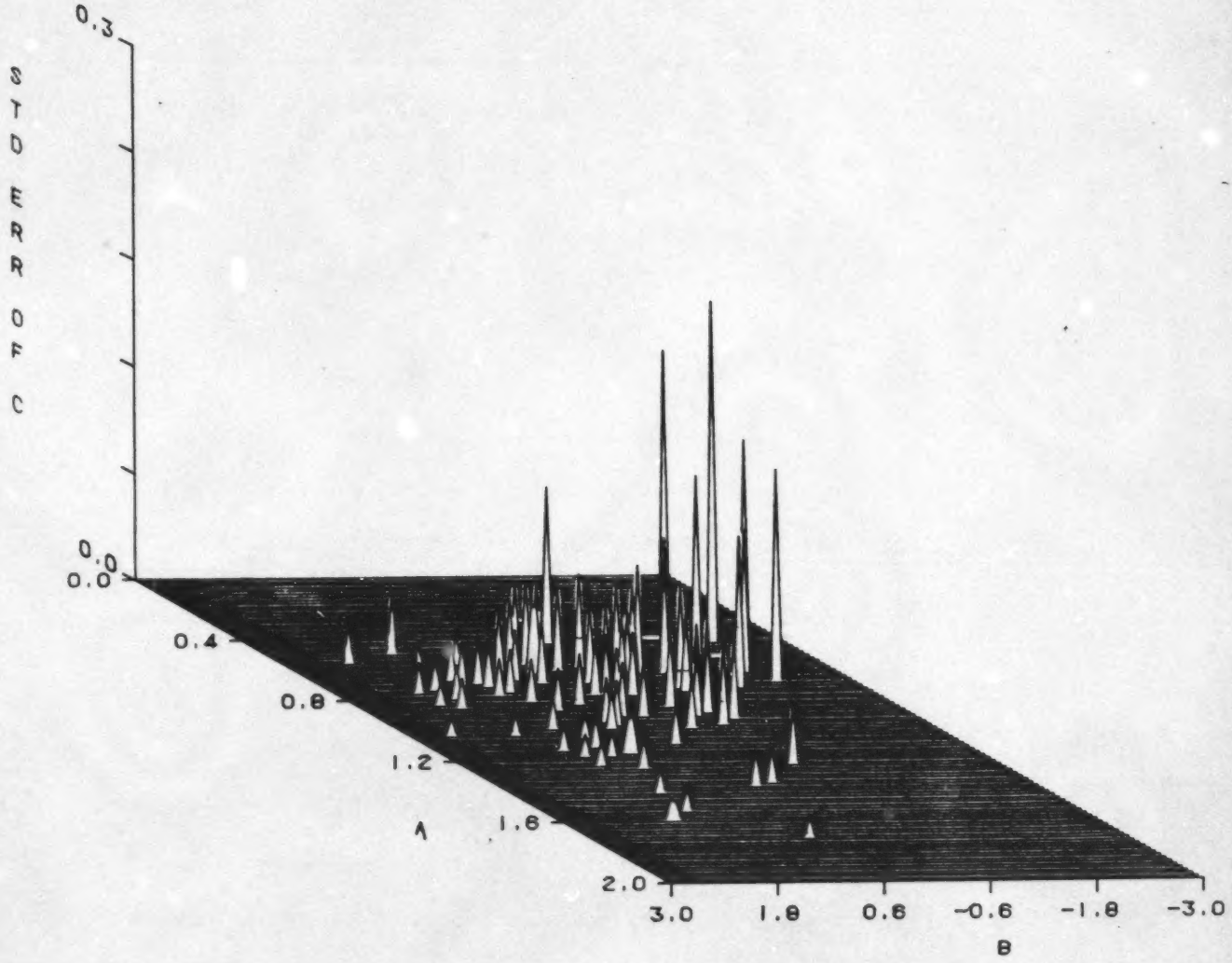

Figure 7. Standard error of $\hat{c}$.

obvious. If the value of c had an important effect on the bias and standard error of the MLE, neighboring items in Figures 1-7 would frequently have quite different biases or standard errors. The fact that neighboring items typically appear very similar in the figures indicates that c typically has a relatively minor effect on their bias and standard error.

Most typically the bias of an MLE is about one-tenth of its standard error. It is very seldom more than a fifth of its standard error.

The effect of the bias for individual item-parameter estimates is thus probably negligible. However, the invariably positive bias in the $\hat{a}$, for example, may have a cumulative effect over many items so that its effect is no longer negligible.

It is just this type of effect that makes the variance across examinees of the MLE of θ a gross overestimate of the true variance of θ across examinees [Lord, Note 1]. An unbiased estimate of σ_θ is derived in the cited reference. Although theoretically possible, it will be more difficult to work out similarly unbiased estimators of equatings and other commonly computed functions of estimated item parameters.

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